Solving Nim Part 1: A Subcase

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August

1 Introduction

Recently, my statistics class at school gave me a peculiar assignment: to build a probability based mock-carnival-game designed to rip off anyone that played it. Though my knee-jerk reaction was to think of slot-machines and dice, I wanted to see if there was a way to create a game allowing for some player strategy, but which would ultimately have a probabilistic outcome. In the deepest recesses of my memory, I recalled some sort of vague game that had something to do with removing objects from various piles. Not recalling anything else about its rules or solution, I decided to fill in the gaps with rules convenient enough to make for a fun carnival game. The basic rules ended up something like this:

- A random number of stacks, would be chosen (between 3 and 5 for practical purposes, but theoretically anything).
- Each stack would be composed of a random number of plastic cups (between 3 and 11 for practical purposes).
- Players would take turns making moves on the configuration, with each move consisting of picking a stack and removing either one or two cups from it.
- The first player who is unable to make a move loses, and their opponent wins.

Little did I know, I had unintentionally created a simplified version of Nim, a famous (usually theoretical) game. In any case, I was soon busy trying to solve my simplified game, and what follows are my results.

2 Winning Condition/Strategy

Consider a row of n stacks of cups. Define the sequence $C = \{C_1, C_2, \ldots, C_n\}$, where C_i is the number of cups in the ith stack. We say that a term in C is a k-residue if it leaves a remainder of k when divided by 3. Furthermore, we denote the number of k-residues in a sequence C as $N_C(k)$. Finally, we call the sequence C "interesting" if both $N_C(1)$ and $N_C(2)$ are even. In any other case, we call the sequence "boring."

Lemma 1: *If C is an interesting sequence, any move on it will yield a boring sequence.*

Proof: Since moves consist of removing either 1 or 2 cups from a given stack, any move is guaranteed to alter the residue of its stack mod 3. Suppose a j-residue gets converted to a k-residue, where $j \neq k$. Since the only possible residues are 0, 1 and 2, at least one of j or k must be either 1 or 2. Observe that our move flips the parity of both $N_C(j)$ and $N_C(k)$. Thus, either the number of 1-residues or the number of 2-residues is no longer even (or both), meaning that the new sequence is boring.

Lemma 2: *If C is a boring sequence, there is always some move that converts it into an interesting sequence.*

Proof: By definition, any boring sequence falls into one of the following three categories:

- $N_C(1)$ is odd and $N_C(2)$ is even. In this case, we can remove one cup from a 1-residue, converting it to a 0-residue and making $N_C(1)$ even.
- $N_C(1)$ is even and $N_C(2)$ is even. Similar to the last case, we can remove two cups from a 2-residue, converting it to a 0-residue and making $N_C(2)$ even.
- Both $N_C(1)$ and $N_C(2)$ are odd. Here we can remove one cup from a 2-residue, converting it to a 1-residue. This decreases $N_C(2)$ by one and increases $N_C(1)$ by one, making both even and resulting in a boring sequence.

It is important to note that for any k, $N_C(k)$ being odd implies that it is nonzero, meaning that at least one k-residue exists. This ensures that all the moves described above can indeed happen.

Theorem: *Player 1 will win if and only if they start with a boring sequence.*

Proof: If Player 1 starts with a boring sequence, they can always force Player 2 into an interesting sequence by Lemma 2. On the next move, Player 2 will always return back a boring sequence to Player 1, due to Lemma 1. This pattern continues, with Player 1 always holding a boring sequence and Player 2 always holding an interesting sequence. Thus, Player 2 will eventually lose, because the losing condition $C = \{0, 0, \ldots, 0\}$ is itself an interesting sequence. Otherwise, if Player 1 starts with an interesting sequence, they will be forced to give a boring sequence to Player 2, and the pattern continues in reverse until Player 1 loses.

3 Probability of Winning

Recall that the number of cups in each stack, i.e. each C_i, is selected randomly from some range. As we have seen, the initial configuration of C entirely determines the outcome of the game. Thus, we can consider the probability with which the random selection of C leads to a win for the first player, i.e. the probability with which C is a boring sequence.

In our case, each C_i was picked uniformly and randomly between 3 and 11, inclusive. As we have seen, a sequence's characterization as boring or interesting only depends on the remainder of its terms mod 3, so we effectively only need to consider sequences consisting of 0, 1 and 2. Our range of 3 to 11 ensures that each remainder (and by extension each sequence of remainders) is equally likely to be picked. Thus, we can simply find the total number of boring remainder-sequences of with length n, and divide by the total number of remainder-sequences of length n, the latter of which is clearly 3^n.

Though there are certainly more algebra-centered proofs, here I will present a more combinatorially-oriented approach.

Lemma 3: *If S is a nonempty set, then the number of odd-sized subsets of S equals the number of even-sized subsets of S.*

Proof: Using the binomial theorem, we see that

$$0 = 0^n = (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} \cdots$$

where the even terms are added and the odd terms are subtracted. Moving all the odd terms to the left side, we see that their sum equals the sum of the even terms, which is equivalent to our lemma.

Our goal is to find the number of boring sequences, i.e. sequences with both an even number of 1s and an even number of 2s. We start by finding the complement: the number of interesting sequences. Define E_1 as the number of sequences with an even number of 1s, and O_1 as the number of sequences with an odd number of 1s. Suppose that there are a total of k zeroes in the sequence: we now choose a subset of the remaining $n-k$ spots to place 1s. By Lemma 3, we see that the number of ways to fill an even amount of 1s equals the number of ways to fill an odd amount of 1s for any choice of k, but with one exception: when $k = n$, all terms in the sequence are 0, leaving no spots to add 1s and resulting in an even count. Thus, E_1 is exactly one greater than O_1. Since $E_1 + O_1 = 3^n$, we see that

$$E_1 = \frac{3^n + 1}{2}.$$

Now, we restrict our attention to these sequences in E_1, and apply similar reasoning for the 2s. Define E_2 to be the number of sequences in E_1 that also have an even number of 2s, and O_2 as the number of sequences in E_1 with an odd number of 2s. Suppose our sequence already has m 1s, where m is an even number. We choose a subset of the remaining $n - m$ terms to fill with 2s. By Lemma 3, the number of ways to fill an even number of 2s equals the number of ways to fill an odd number of 2s for almost any m. The exception cases are are as follows:

- If n is even, then there is a possibility of m equalling n. If so, there will be no space left for the 2s, resulting in an even count. Thus, E_2 will be one greater than O_2.
- If n is odd, then m can never equal n, as m is restricted to be even. Thus, there are no exceptions in this case, and $E_2 = O_2$.

In summary, $E_2 - O_2 = 1$ for even n, and $E_2 - O_2 = 0$ for odd n. We can encode this into a single formula as follows:

$$E_2 - O_2 = \frac{1 + (-1)^n}{2}.$$

By definition, we also have

$$E_2 + O_2 = E_1 = \frac{3^n + 1}{2}.$$

We solve this system of linear equations and finally get

$$E_2 = \frac{3^n + (-1)^n + 2}{4}.$$

By definition, E_2 represents the number of interesting sequences, so the number of boring sequences is

$$3^n - E_2 = \left(\frac{3}{4}\right) 3^n - \frac{2 + (-1)^n}{4}.$$

Finally, we divide out by the total number of sequences, 3^n, and we arrive at our formula for the Player 1's win probability for n stacks:

$$\frac{3}{4} - \frac{2 + (-1)^n}{4 * 3^n}.$$

If made into an actual carnival game, the house could take advantage of the above formula to rig the playing price, ensuring they always have a positive expected monetary gain, *even if the opponent plays perfectly* (which would be unlikely for most carnival-goers). In the next installment of this post, I will discuss how I later tackled problem of solving Nim in its most general case, where players are allowed to remove *any number* of cups from a stack.